

Entanglement of Multipartite Fermionic Coherent States for Pseudo Hermitian Hamiltonians

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Abstract

In this paper the entanglement of multi-qubit fermionic pseudo Hermitian coherent states (FPHCS) described by anticommutative Grassmann numbers is studied. The pseudo-Hermitian versions of the well known maximally entangled pure states such as Bell and GHZ, W and biseparable states are introduced through integrating over the tensor products of FPHCSs with suitable choice of Grassmannian weight functions. Meanwhile as an illustration, the method is applied to tensor product of 2 and 3 qubit pseudo Hermitian systems. Then the measures of concurrence and average entropy are applied to quantify the entanglement of the pseudo two and three qubit states respectively. **Keywords:** Pseudo Hermitian, Entanglement, Pseudo fermionic coherent states, Pseudo Bell states, Pseudo GHZ states, Pseudo Werner states

1 Introduction

Quantum information theory has recently increased its theoretical self-consistency introducing several outstanding results. The most important one has been the achievement that entanglement phenomena of quantum states [1] have been framed in robust theoretical schemes and verified through some experimental tests [2, 3, 4, 5, 6, 7, 8, 9, 10]. In fact entanglement is the most interesting and meanwhile strange feature of quantum physics. The idea of entanglement starts from the apparent conflict between the superposition principle and the nonseparability of the related quantum states. It happens when a state of two or more subsystems of a compound quantum system cannot be factorized into pure local states of the subsystems too. This is equivalent to say that an entangled state could be used to steer a distant particle into one of a set of states, with a certain probability.

Furthermore the recent researches in theoretical physics and quantum optics have revealed the importance of the coherent states. They can be used to encode quantum information on continuous variables [11]. While the entanglement of the bosonic $su(2)$

and $su(1, 1)$ coherent states, as the non orthogonal states which are playing an important role in the quantum cryptography and quantum information processing, has been widely investigated in the references [12, 13, 14, 15, 16, 17, 18, 19], the entanglement properties of multipartite fermionic coherent states are remained as a challenging problem of quantum information theory, even from theoretical point of view [20, 21, 22, 23, 24, 25]. The fermionic coherent states are defined as the eigen-states of the annihilation operator with Grassmannian eigenvalues [26, 27, 28, 29]. On the other hand, the last decade have witnessed a growing interest in non-Hermitian Hamiltonians with real spectra [30, 31, 32, 33, 34, 35, 36, 37]. Considering the results of various numerical studies, Bender and his collaborators [31, 32] found certain examples of one-dimensional non-Hermitian Hamiltonians that possessed real spectra. Because these Hamiltonians were invariant under PT transformations, their spectral properties were linked with their PT-symmetry. Later Mostafazadeh introduced the notion of pseudo-Hermiticity as an alternative possible approach for a non- Hermitian operator to admit a real spectrum [36, 37].

Recently in [23], the entanglement of Grassmannian coherent states for multi-partite n-Level Hermitian systems have been investigated considering tensor product of one mode fermionic coherent states (e.g, $|\theta_1\rangle|\theta_2\rangle$), defined as, $|\theta\rangle = |0\rangle - \theta|1\rangle$, which is presented in terms of standard basis, $(|0\rangle, |1\rangle)$ and anticommuting Grassmann numbers $\theta_i\theta_j = -\theta_j\theta_i$. This rule is justified in the context of quantum field theory, where for example the tensor product of two one-particle states is a two particle state and so on. Then authors found standard maximal entangled Bell, GHZ and W states by integrating over tensor product of two, three and multi-modes fermionic coherent states with proper weight functions. The goal of this paper is extension of the presented method to the pseudo Hermitian systems. For pseudo Hermitian systems, instead of above standard basis we deal with two set of basis $\{|\psi_0\rangle, |\psi_1\rangle\}$ and $\{|\phi_0\rangle, |\phi_1\rangle\}$, which are the eigen-states of H and H^\dagger respectively. Therefore two possible FPHCSs are $|\theta\rangle = |\psi_0\rangle - \theta|\psi_1\rangle$ and $|\tilde{\theta}\rangle = |\phi_0\rangle - \theta|\phi_1\rangle$.

The paper is divided in two main parts. The first part is devoted to construction

of the different families of pseudo Hermitian version of well known maximally entangled pure states such as Bell, GHZ, W and pseudo biseparable states through integrating over the tensor product of FPHCSs of two and three one qubit pseudo Hermitian system with suitable choice of Grassmannian weight functions. Then in section 2 we give a brief introduction about pseudo Hermitian quantum mechanics and in section 3 using the results of generalized Grassmannian pseudo Hermitian coherent state [29] we present the FPHCSs as a special case of generalized Grassmannian pseudo Hermitian coherent states for 2 level system. In section 4 we construct pseudo Hermitian version of Bell states, W and GHZ states. In the second part, section 5, we use the measures of concurrence and average entropy to quantify the entanglement of the pseudo Bell states and GHZ and Werner states respectively and discuss about the results comparing with Hermitian maximal entangled pure states. Finally conclusion is given in section 6.

2 Pseudo-Hermitian Hamiltonians and Biorthonormal Eigenbasis

Intensive study of Schrodinger equation with complex potentials, but with real spectrum, was performed by different methods. The pioneer papers [30, 31, 32, 33, 34, 35, 36, 37], initiated investigation of PT symmetric systems and afterwards more general class of pseudo-Hermitian models was introduced by Mostafazadeh [36, 37]. Following the second approach, let $H : \mathcal{H} \rightarrow \mathcal{H}$ be a linear operator acting in a Hilbert space \mathcal{H} and $\eta : \mathcal{H} \rightarrow \mathcal{H}$ be a linear Hermitian automorphism (invertible transformation). Then the η -pseudo-Hermitian adjoint of H is defined by

$$H^\sharp = \eta^{-1} H^\dagger \eta. \quad (2.1)$$

H is said to be pseudo-Hermitian with respect to η or simply η -pseudo-Hermitian if $H^\sharp = H$. The eigenvalues of pseudo-Hermitian Hamiltonian H are either real or come in

complex-conjugate pairs and the following relations in nondegenerate case hold:

$$H^\dagger = \eta H \eta^{-1}. \quad (2.2)$$

For diagonalizable operators H with discrete spectrum, there exist a complete biorthonormal eigenbasis $\{|\psi_i\rangle, |\phi_i\rangle\}$ such that

$$\begin{aligned} H|\psi_i\rangle &= E_i|\psi_i\rangle, & H^\dagger|\phi_i\rangle &= \bar{E}_i|\phi_i\rangle, \\ \langle\phi_i|\psi_j\rangle &= \delta_{ij}, \end{aligned} \quad (2.3)$$

$$\sum_i |\psi_i\rangle\langle\phi_i| = \sum_i |\phi_i\rangle\langle\psi_i| = I.$$

For a given pseudo-Hermitian H there are infinitely many η satisfying Eq.(2.2). These can however be expressed in terms of a complete biorthonormal basis of H . In non degenerate case the explicit form of η and it's inverse satisfying Eq.(2.2) read

$$\eta = \sum_i |\phi_i\rangle\langle\phi_i|, \quad \eta^{-1} = \sum_i |\psi_i\rangle\langle\psi_i| \quad (2.4)$$

$$|\phi_i\rangle = \eta|\psi_i\rangle, \quad |\psi_i\rangle = \eta^{-1}|\phi_i\rangle. \quad (2.5)$$

Through out the paper, pseudo Hamiltonian H and consequently η and η^{-1} assumed to be in two dimensional Hilbert space.

3 Fermionic Pseudo-Hermitian Coherent States

3.1 Grassmannian variables

The basic properties of Grassmann variables are discussed in Refs.[38, 39, 40, 41] For our purpose, here, we survey the properties of this algebra which is generated by the variables $(\theta_1, \theta_2, \dots, \theta_n)$ satisfying, by definition, the following properties:

$$\begin{aligned} \theta_i \theta_j &= -\theta_j \theta_i \quad , \quad i, j = 1, 2, \dots, n \\ \theta_i^2 &= 0. \end{aligned} \quad (3.6)$$

Analogous rules also apply for the Hermitian conjugate of θ , $\theta^\dagger = \bar{\theta}$, as:

$$\begin{aligned} \bar{\theta}_i \bar{\theta}_j &= -\bar{\theta}_j \bar{\theta}_i \quad , \quad i, j = 1, 2, \dots, n \\ \bar{\theta}_i^2 &= 0. \end{aligned} \quad (3.7)$$

Any linear combination of θ_i with the complex coefficients is called Grassmann number.

In other words, Taylor expansion of a Grassmann function reads

$$g(\theta_1, \theta_2, \dots, \theta_n) = c_0 + \sum_{i=1} c_i \theta_i + \sum_{i,j} c_{i,j} \theta_i \theta_j + \dots,$$

where $c_0, c_i, c_{i,j}, \dots$ are complex numbers. For instance, $\exp(\theta_1 \theta_2) = 1 + \theta_1 \theta_2$. Integration and differentiation over complex Grassmann variables are given by Berezin's rules as :

$$\left\{ \begin{array}{l} \int d\theta f(\theta) = \frac{\partial f(\theta)}{\partial \theta}, \\ \int d\theta = 0, \quad \int d\theta\theta = 1, \quad \int d\bar{\theta} = 0, \quad \int d\bar{\theta}\bar{\theta} = 1, \\ \frac{\partial}{\partial \theta} \theta = 1, \quad \frac{\partial}{\partial \theta} 1 = 0, \quad \frac{\partial}{\partial \theta} \bar{\theta} = 0, \quad \frac{\partial}{\partial \theta} \bar{\theta}\bar{\theta} = 0, \\ \frac{\partial^2}{\partial \theta^2} = 0, \quad \frac{\partial^2}{\partial \theta^2} = 0. \end{array} \right. \quad (3.8)$$

To compute the integral of any function over the Grassmann algebra the following relations are required.

$$\left\{ \begin{array}{l} \theta d\bar{\theta} = -d\bar{\theta}\theta, \quad \bar{\theta} d\theta = -d\theta\bar{\theta} \\ \theta d\theta = -d\theta\theta, \quad \bar{\theta} d\bar{\theta} = -d\bar{\theta}\bar{\theta} \\ d\theta d\bar{\theta} = -d\bar{\theta}d\theta, \quad \theta\bar{\theta} = -\bar{\theta}\theta. \end{array} \right. \quad (3.9)$$

3.2 Coherent States

Following [28, 29], One can construct the pseudo fermionic coherent states for two level pseudo hermitian Hamiltonian. Here we outline the main results . Considering the bi-orthonormality nature of pseudo-Hermitian systems, we can define two pairs of annihilation and creation operators corresponding to the bi-orthonormal eigen-states ($|\psi_i\rangle, |\phi_i\rangle$) respectively as

$$\left\{ \begin{array}{l} b := |\psi_0\rangle\langle\phi_1| + \sqrt{2} |\psi_1\rangle\langle\phi_2|, \\ b^\sharp := \eta^{-1} b^\dagger \eta = |\psi_1\rangle\langle\phi_0| + \sqrt{2} |\psi_2\rangle\langle\phi_1|, \end{array} \right. \quad (3.10)$$

$$\left\{ \begin{array}{l} \tilde{b} = \eta b \eta^{-1} = |\phi_1\rangle\langle\psi_0| + \sqrt{2} |\phi_2\rangle\langle\psi_1|, \\ \tilde{b}^{\sharp'} = \eta'^{-1} b^\dagger \eta = |\phi_0\rangle\langle\psi_1| + \sqrt{2} |\phi_1\rangle\langle\psi_2|, \quad \eta'^{-1} = \eta. \end{array} \right. \quad (3.11)$$

Then it is possible to construct two families of coherent states for two level pseudo Hermitian Grassmannian system in terms of $|\psi_k\rangle$ and $|\phi_k\rangle$. The FPHCSs corresponding

to $|\psi_k\rangle$, $|\phi_k\rangle$ denoted by $|\theta\rangle$ and $|\tilde{\theta}\rangle$ respectively, by definition are the eigen-states of the annihilation operators b and \tilde{b}

$$\begin{cases} b |\theta\rangle = \theta |\theta\rangle, \\ \tilde{b} |\tilde{\theta}\rangle = \theta |\tilde{\theta}\rangle, \end{cases} \quad (3.12)$$

and up to normalization factors are

$$\begin{cases} |\theta\rangle = |\psi_0\rangle - \theta |\psi_1\rangle, \\ |\tilde{\theta}\rangle = |\phi_0\rangle - \theta |\phi_1\rangle. \end{cases} \quad (3.13)$$

The explicit forms of the two families of FPHCS and characteristic of bi-orthonormality of pseudo-Hermitian systems can be exploited for identification of the possible integrals of $|\theta\rangle$ and $|\tilde{\theta}\rangle$, that is $|\theta\rangle\langle\tilde{\theta}|$ and $|\tilde{\theta}\rangle\langle\theta|$, against the measure of $d\bar{\theta} d\theta w(\theta, \bar{\theta})$, which lead to the resolution of identity

$$\int d\bar{\theta} d\theta w(\theta, \bar{\theta}) |\theta\rangle\langle\tilde{\theta}| = \int d\bar{\theta} d\theta w(\theta, \bar{\theta}) |\tilde{\theta}\rangle\langle\theta| = I, \quad (3.14)$$

where $w(\theta, \bar{\theta}) = 1 + \theta\bar{\theta}$. The Eq.(3.14) is called bi-over-completeness relation. To compute weight function we require the following quantization relations between the biorthonormal eigen-states $|\psi_k\rangle$, $|\phi_k\rangle$, ($k = 0, 1$) and Grassmannian variables $\theta, \bar{\theta}$.

$$\begin{cases} \theta |\psi_k\rangle = (-1)^{k-1} |\psi_k\rangle \theta, & \bar{\theta} \langle\psi_k| = (-1)^{k-1} \langle\psi_k| \bar{\theta}, \\ \theta \langle\psi_k| = (-1)^{k-1} \langle\psi_k| \theta, & \bar{\theta} |\psi_k\rangle = (-1)^{k-1} |\psi_k\rangle \bar{\theta}, \\ \theta |\phi_k\rangle = (-1)^{k-1} |\phi_k\rangle \theta, & \bar{\theta} \langle\phi_k| = (-1)^{k-1} \langle\phi_k| \bar{\theta}, \\ \theta \langle\phi_k| = (-1)^{k-1} \langle\phi_k| \theta, & \bar{\theta} |\phi_k\rangle = (-1)^{k-1} |\phi_k\rangle \bar{\theta}. \end{cases} \quad (3.15)$$

The above discussion makes it clear that neither the integral $|\theta\rangle\langle\theta|$, nor the integral of $|\tilde{\theta}\rangle\langle\tilde{\theta}|$, against the measure of $d\bar{\theta} d\theta w(\theta, \bar{\theta})$ normalized:

$$\int d\bar{\theta} d\theta w(\theta, \bar{\theta}) |\theta\rangle\langle\theta| \neq I, \quad \int d\bar{\theta} d\theta w(\theta, \bar{\theta}) |\tilde{\theta}\rangle\langle\tilde{\theta}| \neq I. \quad (3.16)$$

One can show that the fermionic coherent states (3.13) remain coherent for all the times, provided that the time evolution of the initial states managed by Hamiltonian is also an eigen state of lowering operators.

4 Maximal Pseudo Entangled States

Suppose a fermionic system for which the particles can go to the n -mode channels. To this end, we consider tensor product of n one-mode FPHCSs, each one governed by pseudo Hermitian Hamiltonians. For simplicity we consider $n = 2, 3$. The case of arbitrary n is straightforward. Now we introduce the pseudo-Hermitian version of the well known maximally entangled pure two and three qubit states, such as Bell ,GHZ and W states, [42] respectively through integrating over the tensor product of FPHCSs with suitable choice of Grassmannian weight functions.

4.1 Pseudo Bell-like States

Let us start with un-normalized pseudo Hermitian version of standard Bell states,

$$\begin{aligned} |\Psi^\pm\rangle &= \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle), \\ |\Phi^\pm\rangle &= \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) \end{aligned} \quad (4.17)$$

that is

$$|B_1^-\rangle = |\psi_0\rangle |\psi_1\rangle - |\psi_1\rangle |\psi_0\rangle. \quad (4.18)$$

To achieve the above state we consider the tensor product of two one-mode FPHCSs with the same Grassmann numbers as

$$|\theta\rangle |\theta\rangle = |\psi_0\rangle |\psi_0\rangle + \theta (|\psi_0\rangle |\psi_1\rangle - |\psi_1\rangle |\psi_0\rangle). \quad (4.19)$$

As mentioned already, such method is enlightened in the context of quantum field theory. To get the above equation we use the explicit form of, $|\theta\rangle$, from Eq.(3.13) . For the next step, our task is to find the proper weight function $w(\theta)$ such that the integration over Grassmann numbers θ , leads to the Eq.(4.18). To this aim let:

$$\int d\theta w(\theta) |\theta\rangle |\theta\rangle = |B_1^-\rangle, \quad (4.20)$$

putting $w(\theta) = c_0 + c_1\theta$, in the above equation yields $c_0 = 1$ and $c_1 = 0$, then the appropriate weight function takes the form: $w(\theta) = 1$. Considering the tensor product of

$|\theta\rangle|\tilde{\theta}\rangle, |\tilde{\theta}\rangle|\theta\rangle$ and $|\tilde{\theta}\rangle|\tilde{\theta}\rangle$, with $w(\theta) = 1$, it is also possible to construct the other forms of pseudo Bell states as

$$|B_2^-\rangle = \int d\theta |\theta\rangle|\tilde{\theta}\rangle = |\psi_0\rangle|\varphi_1\rangle - |\psi_1\rangle|\varphi_0\rangle, \quad (4.21)$$

So far, we concerned with tensor product of two one-mode FPHCSs with the same Grassmannian numbers, θ , and obtained the pseudo Hermitian versions of $|\Psi^-\rangle$. In order to establish the other pseudo Bell states we need to consider the tensor product of FPHCSs with different Grassmann numbers, i.e.,

$$|\theta_1\rangle|\theta_2\rangle = |\psi_0\rangle|\psi_0\rangle + \theta_2|\psi_0\rangle|\psi_1\rangle - \theta_1|\psi_1\rangle|\psi_0\rangle + \theta_1\theta_2|\psi_1\rangle|\psi_1\rangle, \quad (4.22)$$

in this case the general form of the weight function is $w(\theta_1, \theta_2) = c_0 + c_1\theta_1 + c_2\theta_2 + c_3\theta_1\theta_2$. The task is to find $w(\theta_1, \theta_2)$ such that, in addition to above $|B_i^-\rangle$, the other 3 families of pseudo Bell states are achieved. We denote these 3 families by $|B_i^+\rangle$ and $|B_i'^{\pm}\rangle$. The results summarized in the following table.

4.2 Pseudo GHZ and W States

In the previous subsection using possible tensor product of two one mode FPHCSs we introduced pseudo Bell states. Let us now proceed to construct pseudo version of the following GHZ and W states,

$$\begin{aligned} |GHZ^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|000\rangle \pm |111\rangle), \\ |W\rangle &= \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle), \end{aligned} \quad (4.23)$$

which are used broadly in quantum information theory. To construct 3 qubit pseudo GHZ, we need to consider the tensor product of 3 one mode FPHCSs, with different Grassmann numbers, where they can take following 8 forms

$$\left\{ \begin{array}{l} |\theta_1\rangle|\theta_2\rangle|\theta_3\rangle, |\tilde{\theta}_1\rangle|\theta_2\rangle|\theta_3\rangle, |\theta_1\rangle|\tilde{\theta}_2\rangle|\theta_3\rangle, |\theta_1\rangle|\theta_2\rangle|\tilde{\theta}_3\rangle \\ |\tilde{\theta}_1\rangle|\tilde{\theta}_2\rangle|\theta_3\rangle, |\tilde{\theta}_1\rangle|\theta_2\rangle|\tilde{\theta}_3\rangle, |\theta_1\rangle|\tilde{\theta}_2\rangle|\tilde{\theta}_3\rangle, |\tilde{\theta}_1\rangle|\tilde{\theta}_2\rangle|\tilde{\theta}_3\rangle. \end{array} \right. \quad (4.24)$$

As an illustration we consider the following examples:

$$|G_1^{\pm}\rangle = \int d\theta_1 d\theta_2 d\theta_3 w^{\pm}(\theta_1, \theta_2, \theta_3) |\theta_1\rangle|\theta_2\rangle|\theta_3\rangle = |\psi_0\rangle|\psi_0\rangle|\psi_0\rangle \pm |\psi_1\rangle|\psi_1\rangle|\psi_1\rangle, \quad (4.25)$$

state	FPHCS	weight function	psedo Bell state
$ B_1^\pm\rangle$	$ \theta_1\rangle \theta_2\rangle$	$-(\theta_1 \pm \theta_2)$	$ \psi_0\rangle \psi_1\rangle \pm \psi_1\rangle \psi_0\rangle$
$ B_2^\pm\rangle$	$ \theta_1\rangle \tilde{\theta}_2\rangle$	$-(\theta_1 \pm \theta_2)$	$ \psi_0\rangle \phi_1\rangle \pm \psi_1\rangle \phi_0\rangle$
$ B_3^\pm\rangle$	$ \tilde{\theta}_1\rangle \theta_2\rangle$	$-(\theta_1 \pm \theta_2)$	$ \phi_0\rangle \psi_1\rangle \pm \phi_1\rangle \psi_0\rangle$
$ B_4^\pm\rangle$	$ \tilde{\theta}_1\rangle \tilde{\theta}_2\rangle$	$-(\theta_1 \pm \theta_2)$	$ \phi_0\rangle \phi_1\rangle \pm \phi_1\rangle \phi_0\rangle$
$ B'_1{}^\pm\rangle$	$ \theta_1\rangle \theta_2\rangle$	$-(\theta_1\theta_2 \pm 1)$	$ \psi_0\rangle \psi_0\rangle \pm \psi_1\rangle \psi_1\rangle$
$ B'_2{}^\pm\rangle$	$ \theta_1\rangle \tilde{\theta}_2\rangle$	$-(\theta_1\theta_2 \pm 1)$	$ \psi_0\rangle \phi_0\rangle \pm \psi_1\rangle \phi_1\rangle$
$ B'_3{}^\pm\rangle$	$ \tilde{\theta}_1\rangle \theta_2\rangle$	$-(\theta_1\theta_2 \pm 1)$	$ \phi_0\rangle \psi_0\rangle \pm \phi_1\rangle \psi_1\rangle$
$ B'_4{}^\pm\rangle$	$ \tilde{\theta}_1\rangle \tilde{\theta}_2\rangle$	$-(\theta_1\theta_2 \pm 1)$	$ \phi_0\rangle \phi_0\rangle \pm \phi_1\rangle \phi_1\rangle$
$ B_1^-\rangle$	$ \theta\rangle \theta\rangle$	1	$ \psi_0\rangle \psi_1\rangle - \psi_1\rangle \psi_0\rangle$
$ B_2^-\rangle$	$ \theta\rangle \tilde{\theta}\rangle$	1	$ \psi_0\rangle \phi_1\rangle - \psi_1\rangle \phi_0\rangle$
$ B_3^-\rangle$	$ \tilde{\theta}\rangle \theta\rangle$	1	$ \phi_0\rangle \psi_1\rangle - \phi_1\rangle \psi_0\rangle$
$ B_4^-\rangle$	$ \tilde{\theta}\rangle \tilde{\theta}\rangle$	1	$ \phi_0\rangle \phi_1\rangle - \phi_1\rangle \phi_0\rangle$

Table 1: Unnormalized pseudo Bell states and corresponding weight functions. For example the pseudo Bell state $|\phi_0\rangle|\psi_1\rangle + |\phi_1\rangle|\psi_0\rangle$ can be obtained considering tensor product $|\tilde{\theta}_1\rangle|\theta_2\rangle$ with $w(\theta_1, \theta_2) = (-\theta_1 - \theta_2)$.

where the weight functions are

$$w^\pm(\theta_1, \theta_2, \theta_3) = \theta_3\theta_2\theta_1 \pm 1. \quad (4.26)$$

One can easily check that the appropriate weight function for each of the $|G_i^\pm\rangle$, $i = 1, \dots, 8$ is the same and equals to Eq.(4.26). The results for unnormalized pseudo GHZ states summarized in the the following table To construct the pseudo W states, one can use either the tensor product of FPHCSs with 3 different or the same Grassmann numbers. In the following we introduce one example for each categories, denoted by \mathcal{W} and \mathcal{W}' respectively. For tensor product of the FPHCSs with different Grassmann

state	FPHCS	weight function	pseudo GHZ state
$ G_1^\pm\rangle$	$ \theta_1\rangle \theta_2\rangle \theta_3\rangle$	$\theta_3\theta_2\theta_1 \pm 1$	$ \psi_0\rangle \psi_0\rangle \psi_0\rangle \pm \psi_1\rangle \psi_1\rangle \psi_1\rangle$
$ G_2^\pm\rangle$	$ \tilde{\theta}_1\rangle \theta_2\rangle \theta_3\rangle$	$\theta_3\theta_2\theta_1 \pm 1$	$ \varphi_0\rangle \psi_0\rangle \psi_0\rangle \pm \varphi_1\rangle \psi_1\rangle \psi_1\rangle$
$ G_3^\pm\rangle$	$ \theta_1\rangle \tilde{\theta}_2\rangle \theta_3\rangle$	$\theta_3\theta_2\theta_1 \pm 1$	$ \psi_0\rangle \varphi_0\rangle \psi_0\rangle \pm \psi_1\rangle \varphi_1\rangle \psi_1\rangle$
$ G_4^\pm\rangle$	$ \theta_1\rangle \theta_2\rangle \tilde{\theta}_3\rangle$	$\theta_3\theta_2\theta_1 \pm 1$	$ \psi_0\rangle \psi_0\rangle \varphi_0\rangle \pm \psi_1\rangle \psi_1\rangle \varphi_1\rangle$
$ G_5^\pm\rangle$	$ \tilde{\theta}_1\rangle \tilde{\theta}_2\rangle \theta_3\rangle$	$\theta_3\theta_2\theta_1 \pm 1$	$ \varphi_0\rangle \varphi_0\rangle \psi_0\rangle \pm \varphi_1\rangle \varphi_1\rangle \psi_1\rangle$
$ G_6^\pm\rangle$	$ \tilde{\theta}_1\rangle \theta_2\rangle \tilde{\theta}_3\rangle$	$\theta_3\theta_2\theta_1 \pm 1$	$ \varphi_0\rangle \psi_0\rangle \varphi_0\rangle \pm \varphi_1\rangle \psi_1\rangle \varphi_1\rangle$
$ G_7^\pm\rangle$	$ \theta_1\rangle \tilde{\theta}_2\rangle \tilde{\theta}_3\rangle$	$\theta_3\theta_2\theta_1 \pm 1$	$ \psi_0\rangle \varphi_0\rangle \varphi_0\rangle \pm \psi_0\rangle \varphi_1\rangle \varphi_0\rangle$
$ G_8^\pm\rangle$	$ \tilde{\theta}_1\rangle \tilde{\theta}_2\rangle \tilde{\theta}_3\rangle$	$\theta_3\theta_2\theta_1 \pm 1$	$ \varphi_0\rangle \varphi_0\rangle \varphi_0\rangle \pm \varphi_1\rangle \varphi_1\rangle \varphi_1\rangle$

Table 2: Unnormalized pseudo GHZ states and corresponding weight functions .

numbers we have

$$\begin{aligned}
|\mathcal{W}_1\rangle &= \int d\theta_1 d\theta_2 d\theta_3 w_1(\theta_1, \theta_2, \theta_3) |\theta_1\rangle |\theta_2\rangle |\theta_3\rangle \\
&= |\psi_0\rangle |\psi_0\rangle |\psi_1\rangle + |\psi_0\rangle |\psi_1\rangle |\psi_0\rangle + |\psi_1\rangle |\psi_0\rangle |\psi_0\rangle
\end{aligned} \tag{4.27}$$

where

$$w(\theta_1, \theta_2, \theta_3) = \theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3. \tag{4.28}$$

Similarly for the same Grassmann numbers we get

$$\begin{aligned}
|\mathcal{W}'_1\rangle &= \int d\theta w(\theta) |\theta\rangle |\theta\rangle |\theta\rangle \\
&= -|\psi_0\rangle |\psi_0\rangle |\psi_1\rangle + |\psi_0\rangle |\psi_1\rangle |\psi_0\rangle - |\psi_1\rangle |\psi_0\rangle |\psi_0\rangle,
\end{aligned} \tag{4.29}$$

where the proper weight function is:

$$w'(\theta) = 1. \tag{4.30}$$

As it clears in table *III*, for a given tensor product of three different one mode FPHCSs, e.g, $|\theta_1\rangle |\theta_2\rangle |\theta_3\rangle$ depending on the selection of weight function, there are eight pseudo W states. We emphasized that, although we construct the category \mathcal{W}' in terms of FPHCSs with the same Grassmann numbers, one may also tempt to obtain the same result with the different Grassmann numbers which in turns yield the weight function $w = -\theta_1\theta_2 + \theta_1\theta_3 - \theta_2\theta_3$.

4.3 Pseudo Biseparable States

Here we use FPHCSs to construct pseudo biseparable states. Depending on how one considers bi-partition for a given state, there exists an entanglement in their subsystems partially. For example if a pure state $|ABC\rangle$ involves the three subsystems A,B and C, the partition A may be separable while B,C are entangled. For illustration, let us consider the following examples

$$\left\{ \begin{array}{l} \int d\theta_1 d\theta_2 d\theta_3 (\theta_1 \theta_2 \pm \theta_1 \theta_3) |\theta_1\rangle |\theta_2\rangle |\theta_3\rangle = |\psi_0\rangle_{(1)} \otimes |B_1^\pm\rangle_{(2,3)}, \\ \int d\theta_1 d\theta_2 d\theta_3 (\theta_3 \theta_2 \theta_1 \mp \theta_1) |\theta_1\rangle |\theta_2\rangle |\theta_3\rangle = |\psi_0\rangle_{(1)} \otimes |B_1'^\pm\rangle_{(2,3)}, \\ \int d\theta_1 d\theta_2 d\theta_3 (\theta_1 \theta_2 \mp \theta_3 \theta_2) |\theta_1\rangle |\theta_2\rangle |\theta_3\rangle = |\psi_0\rangle_{(2)} \otimes |B_1^\pm\rangle_{(1,3)}, \end{array} \right. \quad (4.31)$$

where the first two examples show that the partition (2,3) is pseudo Bell state and is separable with respect to partition 1. The same statement holds for partitions (1,3) and 2 in the last example. The above examples make it clear that, we could find different pseudo biseparable states just by considering the integration over tensor product $|\theta_1\rangle |\theta_2\rangle |\theta_3\rangle$ using different weight functions. However one should note that the family W' does not lead to any pseudo biseparable states.

5 Entanglement of Multipartite pseudo Hermitian states

So far, what we have achieved is the constructing of the pseudo Hermitian version of Bell, GHZ and W states. In what follows, we will study the entanglement of pseudo Bell using the measure of concurrence and pseudo GHZ and W states by means of average entropy. To this end let us consider, the following two level PT symmetric [43] or pseudo Hermitian Hamiltonians

$$H_i = \begin{pmatrix} r_i e^{i\beta_i} & s_i \\ t_i & r_i e^{-i\beta_i} \end{pmatrix}, \quad i = 1, 2, 3 \quad (5.32)$$

where index i stands for i^{th} system. We assume that the systems, reside in four and eight dimensional Hilbert space, are governed by $H_1 \otimes H_2$ and $H_1 \otimes H_2 \otimes H_3$ respectively. The bi-orthonormal eigen-sates of H_i , and H_i^\dagger are

$$\begin{cases} |\psi_0\rangle^{(i)} = \frac{1}{\sqrt{2\cos\alpha_i}} (e^{\frac{i\alpha_i}{2}}, e^{-\frac{i\alpha_i}{2}})^T \\ |\psi_1\rangle^{(i)} = \frac{1}{\sqrt{2\cos\alpha_i}} (e^{\frac{-i\alpha_i}{2}}, -e^{\frac{i\alpha_i}{2}})^T \end{cases}, \quad \begin{cases} |\varphi_0\rangle^{(i)} = \frac{1}{\sqrt{2\cos\alpha_i}} (e^{\frac{-i\alpha_i}{2}}, e^{\frac{i\alpha_i}{2}})^T \\ |\varphi_1\rangle^{(i)} = \frac{1}{\sqrt{2\cos\alpha_i}} (e^{\frac{i\alpha_i}{2}}, -e^{-\frac{i\alpha_i}{2}})^T \end{cases}, \quad (5.33)$$

where $\sin\alpha_i = \frac{r_i}{\sqrt{s_i t_i}} \sin\beta_i$, and T denotes the transpose. In the next subsection we first consider the pseudo Bell states.

5.1 Entanglement of pseudo Bell states

It is well known that the entanglement of a two-qubit state $|\psi\rangle$ can be expressed as a function of concurrence [44, 45]

$$\mathcal{C}(|\psi\rangle) \equiv |\langle\psi|\sigma_y \otimes \sigma_y|\psi^*\rangle| \quad (5.34)$$

where σ_y is the y component of the Pauli matrices and $|\psi^*\rangle$ is the complex conjugate of $|\psi\rangle$. Since concurrence itself can also be considered as a measure of entanglement [45], in the following we use it to quantify the entanglement of pseudo Bell states. After normalizing all pseudo Bell states mentioned in table I and recalling the explicit forms of $(|\psi_k\rangle^{(i)})$ and $(|\varphi_k\rangle^{(i)})$, $k = 0, 1$, from Eq.(5.33), the corresponding concurrences take the following forms

$$\begin{cases} \mathcal{C}(|B_1^-\rangle) = \mathcal{C}(|B_4^-\rangle) = |\frac{\cos\alpha_1 \cos\alpha_2}{1-\sin\alpha_1 \sin\alpha_2}|, \\ \mathcal{C}(|B_2^-\rangle) = \mathcal{C}(|B_3^-\rangle) = |\frac{\cos\alpha_1 \cos\alpha_2}{1+\sin\alpha_1 \sin\alpha_2}|, \end{cases} \quad (5.35)$$

where we focused on the third part of table I. Similar discussion can be made for other pseudo Bell states. So the concurrence of $|B_j^-\rangle$ s is a periodic function with respect to the parameters α_1 and α_2 with the period $T = \pi$, that is $\mathcal{C}(\alpha_1, \alpha_2) = \mathcal{C}(\alpha_1 + m\pi, \alpha_2 + m\pi)$, where m belongs to integer numbers. The above equations show that for both cases $\mathcal{C}_{max} = 1$ and $\mathcal{C}_{min} = 0$ appear in $\alpha_1 = \alpha_2 = m\pi$ and $\alpha_1 = \alpha_2 = (2m+1)\frac{\pi}{2}$ respectively.

For the special case $\alpha_1 = \alpha_2 = \alpha$ Eq.(5.35) reads

$$\begin{cases} \mathcal{C}(|B_1^-\rangle) = \mathcal{C}(|B_4^-\rangle) = 1, \\ \mathcal{C}(|B_2^-\rangle) = \mathcal{C}(|B_3^-\rangle) = \frac{\cos^2(\alpha)}{1+\sin^2(\alpha)}. \end{cases} \quad (5.36)$$

It should be no surprise that we obtain $\mathcal{C} = 1$ for $|B_1^-\rangle$ and $|B_4^-\rangle$, independent of parameter α , since for $\alpha_1 = \alpha_2 = \alpha$ these states reduce to standard Bell state $|\Psi^-\rangle$ up to the total phase $e^{-i\pi}$,

$$\begin{cases} |B_1\rangle^- = \frac{|\psi_0\rangle|\psi_1\rangle - |\psi_1\rangle|\psi_0\rangle}{\| |B_1^-\rangle \|} = -\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \\ |B_4\rangle^- = \frac{|\varphi_0\rangle|\varphi_1\rangle - |\varphi_1\rangle|\varphi_0\rangle}{\| |B_4^-\rangle \|} = -\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \end{cases} \quad (5.37)$$

In contrast, the concurrence of $|B_2^-\rangle$ and $|B_3^-\rangle$ varying as a function of α is depicted in Fig.1.

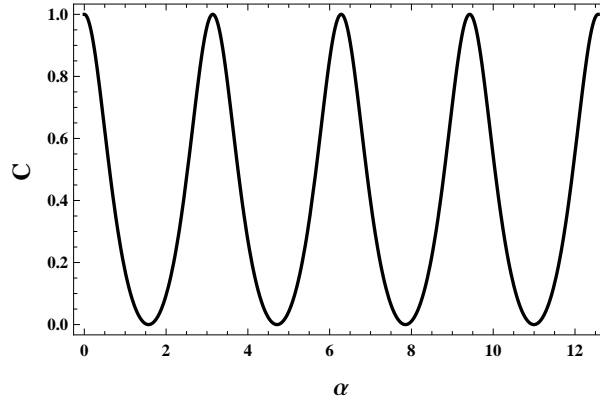


Figure 1: Concurrence of $|B_2^-\rangle$ and $|B_3^-\rangle$ as function of the parameter α .

Simple calculation reveals that for $\alpha_1 = \alpha_2 = \alpha$ the following pseudo Bell states reduce to standard Bell states.

$$\begin{aligned} |B_2'^-\rangle &= |B_3'^-\rangle = |\Psi^+\rangle, \\ |B_1'^+\rangle &= |B_4'^+\rangle = |\Phi^+\rangle, \\ |B_2'^+\rangle &= |B_3'^+\rangle = |\Phi^-\rangle. \end{aligned} \quad (5.38)$$

We consider more special cases which is interesting in dipole interaction decay as follows.

Case a : if $st = r^2 \sin^2 \beta$ then $\mathcal{C}(|B_2^-\rangle) = \mathcal{C}(|B_3^-\rangle) = 0$

Case b : if $r = \frac{\delta}{2}$, $\beta = -\frac{\pi}{2}$, $t = s$, then the Hamiltonian 32 reduce to

$$H_{1,2} = \frac{1}{2} \begin{pmatrix} -i\delta & 2s \\ 2s & i\delta \end{pmatrix}. \quad (5.39)$$

This Hamiltonian arises in interacting two level atom with an electromagnetic field where the real constants δ is the decay rate for the upper and lower levels and the quantity s characterizes the radiation-atom interaction matrix element between the levels described in interaction picture with rotating wave approximation [28, 48, 49]. In this case the concurrence in terms of s and δ is $\mathcal{C}(|B_2^-\rangle) = \mathcal{C}(|B_3^-\rangle) = |\frac{4s^2 - \delta^2}{4s^2 + \delta^2}|$. Since $\sin \alpha = -\frac{\delta}{2s}$, then $4s^2 - \delta^2 \geq 0$ which guarantees the nonnegativity of concurrence. Fig. 2 shows concurrence of $|B_2^-\rangle$ and $|B_3^-\rangle$ for the intervals: $1 \leq s \leq 2$ and $-2 \leq \delta \leq 2$.

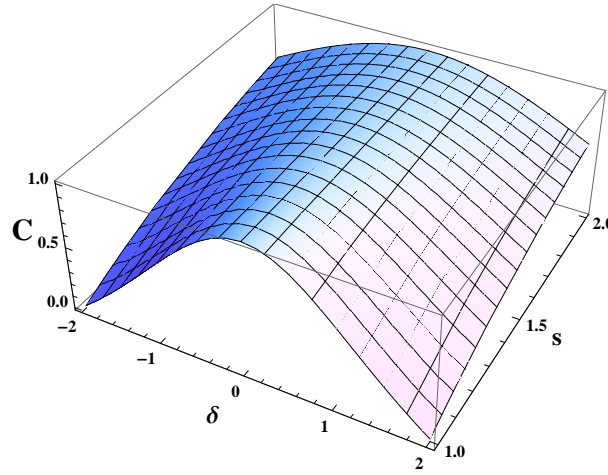


Figure 2: Concurrence of $|B_2^-\rangle$ and $|B_3^-\rangle$ in terms of the parameters δ and s , as it seen for the points $(\delta = 0, s)$ concurrence of these states are equal to one and they are maximally entangled.

5.2 Entanglement of pseudo GHZ and W states

In the previous subsection we studied the entanglement of pseudo Bell states using the measure of concurrence. For the next step we are interested to quantify the entanglement

of pseudo GHZ and W states. The well behavior measure that we shall consider is the average entropy $\langle S_L \rangle$,

$$\langle S_L \rangle = \binom{N}{n}^{-1} \sum_{A_n} S_L^{(A_n; B_{N-n})}, \quad (5.40)$$

which is define via linear entropy S_L [47], as

$$S_L^{(A_n; B_{N-n})} = \frac{d}{d-1} (1 - \text{Tr}_{A_n} [\rho_{A_n}]^2), \quad \rho_{A_n} = \text{Tr}_{B_{N-n}} [\rho], \quad (5.41)$$

where, $d = \min\{2^n, 2^{N-n}\}$, is the dimension of the reduced density matrix ρ_{A_n} . It should be recall that although the linear entropy and von Neumann entropy [46] are similar measures of the mixedness of a state, the linear entropy is easier to calculate because it does not require the diagonalization of the density matrix. The linear entropy can range between zero, corresponding to a completely pure state, and, 1 corresponding to a completely mixed state. Based on the measure of average entropy, as examples, we shall investigated the entanglement of the normalized $|G_1^+\rangle$, $|\mathcal{W}_7^{(+,+,+)}\rangle$ and $|\mathcal{W}_6^{(-,+, -)}\rangle$ (for simplicity denoted by W_7 and W_6 respectively) as

$$\begin{aligned} |G_1^+\rangle &= \frac{\{|\varphi_0\rangle|\varphi_0\rangle|\varphi_0\rangle \pm |\varphi_1\rangle|\varphi_1\rangle|\varphi_1\rangle\}}{\| |G_1^+\rangle \|} \\ |\mathcal{W}_7\rangle &= \frac{|\psi_0\rangle|\varphi_0\rangle|\varphi_1\rangle + |\psi_0\rangle|\varphi_1\rangle|\varphi_0\rangle + |\psi_1\rangle|\varphi_0\rangle|\varphi_0\rangle}{\| |\mathcal{W}_7\rangle \|} \\ |\mathcal{W}_6\rangle &= \frac{-|\varphi_0\rangle|\psi_0\rangle|\varphi_1\rangle + |\varphi_0\rangle|\psi_1\rangle|\varphi_0\rangle - |\varphi_1\rangle|\psi_0\rangle|\varphi_0\rangle}{\| |\mathcal{W}_6\rangle \|} \end{aligned} \quad (5.42)$$

Regarding the definition of the Eq.(5.40) the average entropy of the normalized $|G_1^+\rangle$ is given as:

$$\langle S_L \rangle_{(G_1^+)} = \frac{1}{6} (5 + \cos 2\alpha_2 - 2 \sin^2 \alpha_1 (1 + \cos^2 \alpha_3 \sin^2 \alpha_2) + (\cos 2\alpha_1 \cos 2\alpha_2 - 3) \sin^2 \alpha_3) \quad (5.43)$$

Straightforward calculations reveal that, the average entropy of all of the pseudo GHZ states are the same and equal to Eq.(5.43). As before, let us consider the quantum states with $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, which yields

$$\langle S_L \rangle_{(G_1^+)} = \frac{1}{2} \cos^4 \alpha (3 - \cos 2\alpha). \quad (5.44)$$

Fig. 3 shows the average entropy of the $|G_1^+\rangle$, in terms of the parameter α . The maximum and minimum value of average entropy for pseudo GHZ states occur at the points $\alpha = k\pi$ and $\alpha = (2k + 1)\frac{\pi}{2}$ respectively.

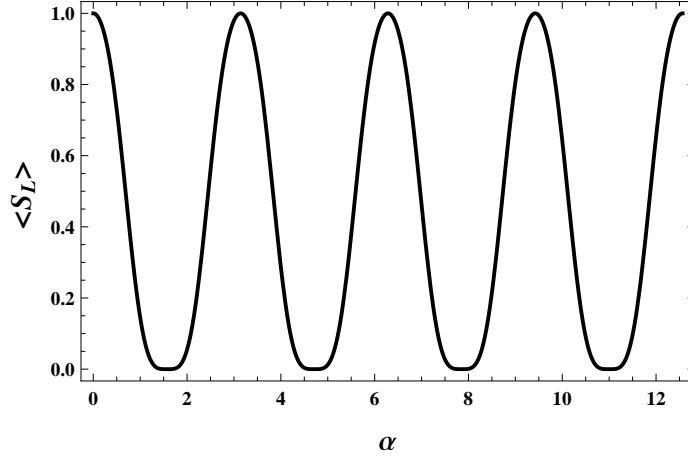


Figure 3: The average entropy of all pseudo GHZ states, verses parameter α .

As another example, we shall consider the normalized $|\mathcal{W}_7\rangle$, where its average entropy for the cases of different and identical α_i respectively are

$$\langle S_L \rangle_{(\mathcal{W}_7)} = \quad (5.45)$$

$$\frac{2(\cos 2\alpha_1 + \cos 2\alpha_2 + 2) \cos 2\alpha_3 + \cos 2(\alpha_1 - \alpha_2) + \cos 2(\alpha_1 + \alpha_2) + 4 \cos 2\alpha_1 + 4 \cos 2\alpha_2 + 6}{3(2 \sin \alpha_2 \sin \alpha_3 - 2 \sin \alpha_1 (\sin \alpha_2 + \sin \alpha_3) + 3)^2}$$

$$\langle S_L \rangle_{(\mathcal{W}_7)} = \frac{8 \cos^4 \alpha}{(\cos 2\alpha + 2)^2}. \quad (5.46)$$

Fig. 4 shows that the average entropy of $|\mathcal{W}_7\rangle$, for identical case is bounded by the following values

$$0 \leq \langle S_L(\alpha) \rangle_{(\mathcal{W}_7)} \leq \frac{8}{9},$$

where the upper and lower bounds appear at $\alpha = k\pi$ and $\alpha_k = (2k + 1)\frac{\pi}{2}$ respectively.

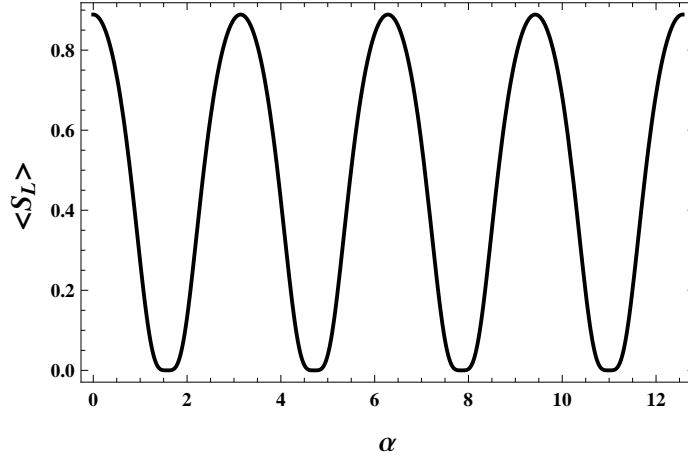


Figure 4: The average entropy of $|\mathcal{W}_7\rangle$, as a function of α . The lower bound 0 and upper bound $\frac{8}{9}$ correspond to separable and maximal entangled pseudo $|\mathcal{W}_7\rangle$ states respectively.

Finally as the last example let us study the average entropy of the normalized $|\mathcal{W}_6\rangle$.

To this end, considering the Eq.(5.40) we deduce the following expression for $\langle S_L \rangle_{(\mathcal{W}_6)}$

$$\langle S_L \rangle_{(\mathcal{W}_6)} = \frac{2(\cos 2\alpha_1 + \cos 2\alpha_2 + 2) \cos 2\alpha_3 + \cos 2(\alpha_1 - \alpha_2) + \cos 2(\alpha_1 + \alpha_2) + 4 \cos 2\alpha_1 + 4 \cos 2\alpha_2 + 6}{3(2 \sin \alpha_2 \sin \alpha_3 + 2 \sin \alpha_1 (\sin \alpha_2 + \sin \alpha_3) + 3)^2} \quad (5.47)$$

It is easy to check that for the case of $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ the above equation reduce to

$$\langle S_L \rangle_{(\mathcal{W}_6)} = \frac{8 \cos^4 \alpha}{9(\cos 2\alpha - 2)^2}. \quad (5.48)$$

Fig. 5, shows the $\langle S_L \rangle_{(\mathcal{W}_6)}$ in terms of the parameter α and like the previous cases, the maximum value of the average entropy of the $|\mathcal{W}_6\rangle$ is exactly the same as that of the entangled states described in standard Hermitian Hamiltonian. The method presented can also be extend to mulitipartide n level systems.

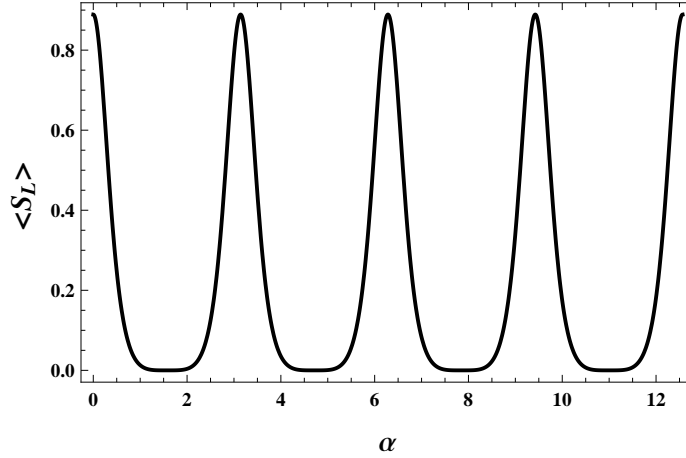


Figure 5: Average entropy of $|W_6\rangle$ as a function of α .

6 Conclusion

In conclusion we have constructed the pseudo-Hermitian version of the well known maximally entangled pure states such as Bell GHZ, W, and biseparable states, by integrating over tensor product of one-mode FPHCSs and using suitable Grassmannian weight function. Meanwhile to clarify the issue, we explicitly consider the bi-orthonormal eigen-states of pseudo Hermitian Hamiltonian which appears in interacting two level atom with an electromagnetic field. In order to quantify the entanglement of aforementioned pseudo states, we used concurrence measure and average linear entropy for two qubit (pseudo Bell states) and 3 qubit (pseudo GHZ and W) respectively. It is found that for $\alpha_1 = \alpha_2 = \alpha$ pseudo Bell states $|B_1^-\rangle$ and $|B_4^-\rangle$ up to the total phase $e^{-i\pi}$, are the same as standard Bell state $|\Psi^-\rangle$. Similarly $|B_2^-\rangle$ and $|B_3^-\rangle$ are the same as $|\Psi^-\rangle$ and $|B_1^+\rangle$ and $|B_4^+\rangle$ are equal to $|\Phi^+\rangle$ and $|B_2^+\rangle$ and $|B_3^+\rangle$ reduce to $|\Phi^-\rangle$.

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state	FPHCS	weight function	pseudo W state
$ W_1^{(i)}\rangle$	$ \theta_1\rangle \theta_2\rangle \theta_3\rangle$	$\pm\theta_1\theta_2 \pm \theta_1\theta_3 \pm \theta_2\theta_3$	$\pm \psi_0\rangle \psi_0\rangle \psi_1\rangle \pm \psi_0\rangle \psi_1\rangle \psi_0\rangle \pm \psi_1\rangle \psi_0\rangle \psi_0\rangle$
$ W_2^{(i)}\rangle$	$ \tilde{\theta}_1\rangle \theta_2\rangle \theta_3\rangle$	$\pm\theta_1\theta_2 \pm \theta_1\theta_3 \pm \theta_2\theta_3$	$\pm \varphi_0\rangle \psi_0\rangle \psi_1\rangle \pm \varphi_0\rangle \psi_1\rangle \psi_0\rangle \pm \varphi_1\rangle \psi_0\rangle \psi_0\rangle$
$ W_3^{(i)}\rangle$	$ \theta_1\rangle \tilde{\theta}_2\rangle \theta_3\rangle$	$\pm\theta_1\theta_2 \pm \theta_1\theta_3 \pm \theta_2\theta_3$	$\pm \psi_0\rangle \varphi_0\rangle \psi_1\rangle \pm \psi_0\rangle \varphi_1\rangle \psi_0\rangle \pm \psi_1\rangle \varphi_0\rangle \psi_0\rangle$
$ W_4^{(i)}\rangle$	$ \theta_1\rangle \theta_2\rangle \tilde{\theta}_3\rangle$	$\pm\theta_1\theta_2 \pm \theta_1\theta_3 \pm \theta_2\theta_3$	$\pm \psi_0\rangle \psi_0\rangle \varphi_1\rangle \pm \psi_0\rangle \psi_1\rangle \varphi_0\rangle \pm \psi_1\rangle \psi_0\rangle \varphi_0\rangle$
$ W_5^{(i)}\rangle$	$ \tilde{\theta}_1\rangle \tilde{\theta}_2\rangle \theta_3\rangle$	$\pm\theta_1\theta_2 \pm \theta_1\theta_3 \pm \theta_2\theta_3$	$\pm \varphi_0\rangle \varphi_0\rangle \psi_1\rangle \pm \varphi_0\rangle \varphi_1\rangle \psi_0\rangle \pm \varphi_1\rangle \varphi_0\rangle \psi_0\rangle$
$ W_6^{(i)}\rangle$	$ \tilde{\theta}_1\rangle \theta_2\rangle \tilde{\theta}_3\rangle$	$\pm\theta_1\theta_2 \pm \theta_1\theta_3 \pm \theta_2\theta_3$	$\pm \varphi_0\rangle \psi_0\rangle \varphi_1\rangle \pm \varphi_0\rangle \psi_1\rangle \varphi_0\rangle \pm \varphi_1\rangle \psi_0\rangle \varphi_0\rangle$
$ W_7^{(i)}\rangle$	$ \theta_1\rangle \tilde{\theta}_2\rangle \tilde{\theta}_3\rangle$	$\pm\theta_1\theta_2 \pm \theta_1\theta_3 \pm \theta_2\theta_3$	$\pm \psi_0\rangle \varphi_0\rangle \varphi_1\rangle \pm \psi_0\rangle \varphi_1\rangle \varphi_0\rangle \pm \psi_1\rangle \varphi_0\rangle \varphi_0\rangle$
$ W_8^{(i)}\rangle$	$ \tilde{\theta}_1\rangle \tilde{\theta}_2\rangle \tilde{\theta}_3\rangle$	$\pm\theta_1\theta_2 \pm \theta_1\theta_3 \pm \theta_2\theta_3$	$\pm \varphi_0\rangle \varphi_0\rangle \varphi_1\rangle \pm \varphi_0\rangle \varphi_1\rangle \varphi_0\rangle \pm \varphi_1\rangle \varphi_0\rangle \varphi_0\rangle$
$ W'_1\rangle$	$ \theta\rangle \theta\rangle \theta\rangle$	1	$- \psi_0\rangle \psi_0\rangle \psi_1\rangle + \psi_0\rangle \psi_1\rangle \psi_0\rangle - \psi_1\rangle \psi_0\rangle \psi_0\rangle$
$ W'_2\rangle$	$ \tilde{\theta}\rangle \theta\rangle \theta\rangle$	1	$- \varphi_0\rangle \psi_0\rangle \psi_1\rangle + \varphi_0\rangle \psi_1\rangle \psi_0\rangle - \varphi_1\rangle \psi_0\rangle \psi_0\rangle$
$ W'_3\rangle$	$ \theta\rangle \tilde{\theta}\rangle \theta\rangle$	1	$- \psi_0\rangle \varphi_0\rangle \psi_1\rangle + \psi_0\rangle \varphi_1\rangle \psi_0\rangle - \psi_1\rangle \varphi_0\rangle \psi_0\rangle$
$ W'_4\rangle$	$ \theta\rangle \theta\rangle \tilde{\theta}\rangle$	1	$- \psi_0\rangle \psi_0\rangle \varphi_1\rangle + \psi_0\rangle \psi_1\rangle \varphi_0\rangle - \psi_1\rangle \psi_0\rangle \varphi_0\rangle$
$ W'_5\rangle$	$ \tilde{\theta}\rangle \tilde{\theta}\rangle \theta\rangle$	1	$- \varphi_0\rangle \varphi_0\rangle \psi_1\rangle + \varphi_0\rangle \varphi_1\rangle \psi_0\rangle - \varphi_1\rangle \varphi_0\rangle \psi_0\rangle$
$ W'_6\rangle$	$ \tilde{\theta}\rangle \theta\rangle \tilde{\theta}\rangle$	1	$- \varphi_0\rangle \psi_0\rangle \varphi_1\rangle + \varphi_0\rangle \psi_1\rangle \varphi_0\rangle - \varphi_1\rangle \psi_0\rangle \varphi_0\rangle$
$ W'_7\rangle$	$ \theta\rangle \tilde{\theta}\rangle \tilde{\theta}\rangle$	1	$- \psi_0\rangle \varphi_0\rangle \varphi_1\rangle + \psi_0\rangle \varphi_1\rangle \varphi_0\rangle - \psi_1\rangle \varphi_0\rangle \varphi_0\rangle$
$ W'_8\rangle$	$ \tilde{\theta}\rangle \tilde{\theta}\rangle \tilde{\theta}\rangle$	1	$- \varphi_0\rangle \varphi_0\rangle \varphi_1\rangle + \varphi_0\rangle \varphi_1\rangle \varphi_0\rangle - \varphi_1\rangle \varphi_0\rangle \varphi_0\rangle$

Table 3: Unnormalized pseudo W states and corresponding weight functions. The upper index (i) refers to set of symbols $\{(+,+,+), (+,+,-), \dots (-,-,-)\}$, addressing to the set of weight functions $\{\theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3, \theta_1\theta_2 + \theta_1\theta_3 - \theta_2\theta_3, -\theta_1\theta_2 - \theta_1\theta_3 - \theta_2\theta_3\}$ respectively.